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Low-temperature properties of the generalized two-chain quantum spin model

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Abstract. We consider an exactly solvable two-chain quantum spin-$\frac{1}{2}$ model in a generalized form with enhanced spin frustration. This model is exactly diagonalized via Bethe's ansatz. The low-temperature specific heat of the system is obtained with and without a magnetic field, using the thermodynamic Bethe ansatz equations. We also calculate the magnetic susceptibility in a sufficiently weak field, yielding typical logarithmic corrections. The spin frustration affects only the amplitudes in the magnetic susceptibility and the specific heat. This extends the previous results for a simple two-chain quantum spin model to the generalized one.

1. Introduction

The discovery of numerous high-temperature superconductors has renewed the interest in low-dimensional systems. It is known to be common to all compounds exhibiting a high T_c that high-T_c superconductivity is strongly related to the layers containing copper and oxygen atoms. Some theorists have approached this problem using a one-dimensional chain of copper and oxygen atoms for simplicity, even though the two-dimensional Hubbard model is more relevant. The idea of the one-dimensional chain is extended to copper oxide ladders which have structures of pairs of copper oxide chains linked by additional oxygen atoms between coppers [1, 2]. Recently, a few compounds have been realized experimentally with a ladder structure [3] and one of these compounds, $Sr_{0.4}Ca_{13.6}Cu_{24}O_{41.18}$ is reported to have superconductivity [4]. These compounds can be mapped to spin-$\frac{1}{2}$ antiferromagnetic ladders with frustration only if considering the magnetic properties. Understanding the ground state for the spin-frustrated system is one of the most interesting issues in the magnetic properties of solids. Phosphates $VO(HPO_4)4H_2O$ [5] are one of the realizations of the two-chain quantum spin model and the two-plane quantum Hall effect [6] also shows some properties of the two-chain quantum spin model.

An exactly solvable multi-chain quantum spin model has been constructed by using the quantum inverse scattering method, and the thermodynamics of the model has been discussed via the Bethe ansatz method in [7–9, 11]. The transfer matrix $\hat{T}(\lambda)$ for the multi-chain spin-$\frac{1}{2}$ model is expressed by a product of the transfer matrices for the typical Heisenberg model with shifted spectral parameters. For instance, $\hat{T}(\lambda) = \hat{t}(\lambda+\kappa)\,\hat{t}(\lambda)$ for a two-chain case, where $\hat{t}$ is the transfer matrix of a single chain and κ denotes the interchain coupling. The corresponding Hamiltonian contains the terms breaking P and T symmetry which is responsible for the chiral behaviour in the thermodynamic properties.

In this paper, we construct an exactly solvable two-chain quantum spin model in a generalized form, using the generalized transfer matrix of $\hat{T}(\lambda) = \hat{t}^{\alpha}(\lambda+\kappa)\,\hat{t}^{\beta}(\lambda)$, where

α and β are constants. This transfer matrix also yields an integrable Hamiltonian by construction, however, the Hamiltonian is more frustrated by arbitrary α and β. In section 2, we present the generalized two-chain spin Hamiltonian and the Bethe ansatz equations. In section 3, we obtain the Bethe ansatz equation in the limit of zero temperature and a weak but finite magnetic field, which means that the temperature goes to zero faster than the magnetic field. We calculate the magnetic susceptibility χ_s and the linear coefficient of specific heat γ in a magnetic field. On the other hand, in section 4, we discuss the limit of zero field and a low but finite temperature in which the magnetic field goes to zero faster than the temperature, and γ is obtained at zero magnetic field. Concluding remarks follow in section 5.

2. Model and Bethe ansatz equations

We consider the Hamiltonian for a two-chain spin system

$$\mathcal{H} = \frac{\alpha+\beta}{1+\kappa^2}\sum_{n=1}^{N}(\vec{S}_{2n-1}\cdot\vec{S}_{2n}+\vec{S}_{2n}\cdot\vec{S}_{2n+1}) + \frac{\kappa^2}{1+\kappa^2}\sum_{n=1}^{N}(\alpha\vec{S}_{2n-1}\cdot\vec{S}_{2n+1}+\beta\vec{S}_{2n}\cdot\vec{S}_{2n+2}) + \frac{2\kappa}{1+\kappa^2}\sum_{n=1}^{N}(\alpha\vec{S}_{2n-1}-\beta\vec{S}_{2n+2})\cdot(\vec{S}_{2n+1}\times\vec{S}_{2n}) - H\sum_{n=1}^{N}\left(S^z_{2n-1}+S^z_{2n}\right) - E_f \tag{2.1}$$

where $\vec{S}_{2n-1}$ ($\vec{S}_{2n}$) is spin $\frac{1}{2}$ operator at site $2n-1$ ($2n$) in the first (second) chain, N is the number of spins on each chain, κ is related to the interchain coupling, and E_f is the energy of the ferromagnetic state, i.e. $E_f = (\alpha+\beta)\left(2+\kappa^2\right)/\left[4\left(1+\kappa^2\right)\right]$. α and β are the arbitrary constants and H is the external magnetic field. Note that the model is reduced to a single Heisenberg chain of $2N$ spins with exchange coupling $\alpha+\beta$ when $\kappa = 0$.

The third term in the Hamiltonian has an unusual form and it is not avoidable for the integrability of the two-chain spin system. When $\alpha = \beta$, the third term breaks both T and P symmetries, while conserving TP symmetry and the neighbouring triangular spin sets have chiralities of different signs for the antiferromagnetic case [9, 10]. Arbitrary α and β can enhance the spin frustration of the Hamiltonian. It is especially interesting to consider that $\alpha > 0$, $\beta < 0$ and $\alpha+\beta > 0$. This situation describes an antiferromagnetic interchain interaction, while the spins in the first (second) chain tend to interact antiferromagnetically (ferromagnetically) with each other.

The Hamiltonian in (2.1) was obtained by the logarithmic derivative of the transfer matrix $\hat{T}$ as usual. The transfer matrix of the system is constructed by using the quantum inverse scattering method as

$$\hat{T}(\lambda) = \hat{t}^{\alpha}(\lambda+\kappa)\,\hat{t}^{\beta}(\lambda) \tag{2.2}$$

where $\hat{t}$ is the standard transfer matrix of the single chain of spin $\frac{1}{2}$ and λ is the spectral parameter. The diagonalization of the transfer matrix $\hat{T}$ requires that the spin rapidities λ_j, satisfy the discrete Bethe ansatz equations

$$\left(\frac{\lambda_j-\kappa-i}{\lambda_j-\kappa+i}\right)^N\left(\frac{\lambda_j+\kappa-i}{\lambda_j+\kappa+i}\right)^N = -\prod_{k=1}^{M}\frac{\lambda_j-\lambda_k-2i}{\lambda_j-\lambda_k+2i} \qquad j = 1, 2, \ldots, M \tag{2.3}$$

with M denoting the total number of down spins in two chains of the system. Periodic boundary conditions have been imposed to derive (2.3) and the λ values are, in general, complex numbers. These discrete Bethe ansatz equations do not depend on the values of

α and β, since the eigenfunctions of the transfer matrix do not change with α and β by construction. The eigenvalues, however, do change due to α and β. The solutions of (2.3) determine the energy eigenvalues,

$$E = -2\sum_{j=1}^{M}\left[\frac{\alpha}{(\lambda_j-\kappa)^2+1}+\frac{\beta}{(\lambda_j+\kappa)^2+1}\right] - H(N-M). \tag{2.4}$$

For large N, we have the so-called string solutions such as $\lambda^{j}_{n,\alpha} = \lambda_{n,\alpha} + \mathrm{i}(n+1-2j)$, $j = 1, 2, \ldots, n$ where $\lambda_{n,\alpha}$ is real and $n = 1, 2, \ldots, \infty$. There are M_n strings of length n, labelled by the index α, hence, the total number of electrons with spin down is $M = \sum_{n=1}^{\infty} nM_n$.

In the thermodynamic limit the usual distribution densities $\rho_n(\lambda)$ for the occupied real parameters $\lambda_{n,\alpha}$ and similarly the corresponding 'hole' distribution functions $\rho_{n,h}(\lambda)$ for the unoccupied rapidities [12, 13] are introduced. These distribution functions are related by the following integral equations:

$$\rho_{n,h}(\lambda) + \sum_{m=1}^{\infty} A_{n,m} * \rho_m = \tfrac{1}{2}[a_n(\lambda-\kappa) + a_n(\lambda+\kappa)] \tag{2.5}$$

where $*$ denotes convolution. The integration kernels $A_{nm}(\lambda)$ and $a_n(\lambda)$ are the Fourier transforms of

$$\begin{aligned}\tilde{A}_{n,m}(\omega) &= \mathrm{e}^{\mathrm{i}\lambda\omega}\coth|\omega|\big[\mathrm{e}^{-(n-m)|\omega|} - \mathrm{e}^{-(n+m)|\omega|}\big]\\ \tilde{a}_n(\omega) &= \mathrm{e}^{-n|\omega|}.\end{aligned} \tag{2.6}$$

The energy per spin is then given by

$$\frac{E}{2N} = -2\pi\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,[\alpha a_n(\lambda-\kappa)+\beta a_n(\lambda+\kappa)]\rho_n - H\left[\tfrac{1}{2} - \sum_{n=1}^{\infty} n\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\rho_n\right]. \tag{2.7}$$

As usual, the rapidities obey Fermi statistics so that the entropy of the system can be written as

$$\begin{aligned}\frac{S}{2N} = \sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\lambda\,\{&[\rho_n(\lambda)+\rho_{n,h}(\lambda)]\ln[\rho_n(\lambda)+\rho_{n,h}(\lambda)]\\ &-\rho_n(\lambda)\ln\rho_n(\lambda) - \rho_{n,h}(\lambda)\ln\rho_{n,h}(\lambda)\}\end{aligned} \tag{2.8}$$

and the density functions are obtained by minimizing the thermodynamic potential $F = E - TS$ with respect to the distribution functions, subject to the constraints in (2.5). It is convenient to introduce $\eta_n = \rho_{n,h}/\rho_n$ and in terms of these functions we have the thermodynamic Bethe ansatz equations

$$\ln[1+\eta_n(\lambda)] = \frac{nH}{T} - \frac{2\pi}{T}[\alpha a_n(\lambda-\kappa)+\beta a_n(\lambda+\kappa)] + \sum_{m=1}^{\infty} A_{n,m} * \ln\left(1+\eta_m^{-1}\right). \tag{2.9}$$

Equivalently, η_n satisfy the following nonlinearly coupled integral equations:

$$\ln\eta_n = -\frac{2\pi}{T}[\alpha G_0(\lambda-\kappa)+\beta G_0(\lambda+\kappa)]\delta_{n,1} + G_0 * \ln(1+\eta_{n-1})(1+\eta_{n+1}) \tag{2.10}$$

where $G_n(\lambda)$ is defined by

$$G_n(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\omega\,\frac{\mathrm{e}^{-n|\omega|}}{2\cosh|\omega|}\mathrm{e}^{-\mathrm{i}\omega\lambda}. \tag{2.11}$$

The free energy per spin is then given by

$$\frac{F(T,H)}{2N} - \frac{F(0,0)}{2N} = -\frac{T}{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda\, [G_0(\lambda-\kappa) + G_0(\lambda+\kappa)] \ln[1+\eta_1(\lambda)] \tag{2.12}$$

where $F(0,0)/2N = -(\alpha+\beta)\pi[G_1(0) + G_1(2\kappa)]$ with $2\pi G_1(0)$ being $\ln 2$. Note that for $\kappa = 0$, the ground-state energy per spin reduces to $-(\alpha+\beta)\ln 2$ as expected from the conventional Heisenberg chain of $2N$ spins.

3. Zero-temperature limit in field

The ground-state equations are recovered in the limit $T \to 0$. It is convenient to introduce the following energy potentials for the excitations, $\epsilon_n(\lambda) = T \ln \eta_n(\lambda)$. These energy potentials are further separated into their positive and negative parts, ϵ_n^+ and ϵ_n^-, respectively. In the $T \to 0$ limit it can be shown that $\epsilon_{n\geqslant 2}(\lambda) > 0$ and therefore $\epsilon_{n\geqslant 2}^-(\lambda) = 0$ for all λ-values after some algebra [13]. Hence the ground state of the system is described only by ϵ_1. Fourier transforming (2.9) for $n = 1$, the equation describing the ground state of the system is given by

$$\epsilon_1(\lambda) = \tfrac{1}{2}H - 2\pi[\alpha G_0(\lambda-\kappa) + \beta G_0(\lambda+\kappa)] + G_1 * T \ln\left(1 + \mathrm{e}^{\epsilon_1/T}\right). \tag{3.1}$$

In the case of $\alpha = \beta$ or $\kappa = 0$, it is well known that $\epsilon_1(\lambda)$ is a symmetric and monotonically increasing function for positive arguments [13]. Generally, however, it is asymmetric and has very different behaviour depending upon α, β and κ. In a weak magnetic field with $\alpha > 0$, $\beta < 0$, $\alpha + \beta > 0$ and $0 < \kappa < (1/\pi)\ln|\alpha/\beta|$, the driving terms are simply shifted to positive λ compared to those of $\alpha = \beta$ or $\kappa = 0$. This shift breaks the symmetry of $\epsilon_1(\lambda)$ in λ. If $\kappa > (1/\pi)\ln|\alpha/\beta|$, on the other hand, then the driving terms generate a new maximum around $\lambda \sim -\kappa$ and a minimum around $\lambda \sim \kappa$ which yield totally different behaviours in $\epsilon_1(\lambda)$ and cannot be treated analytically. In this section, let us obtain analytically the magnetic susceptibility and the specific heat in a weak magnetic field ($0 < H \ll (\alpha+\beta)/\left(1+\kappa^2\right)$) by solving this integral equation analytically for $\alpha > 0$, $\beta < 0$, $\alpha + \beta > 0$ and $0 < \kappa < (1/\pi)\ln|\alpha/\beta|$.

3.1. Magnetic susceptibility at $T = 0$

Let us denote $\epsilon_1(\lambda)$ at $T = 0$ as $\epsilon_1^{(0)}(\lambda)$. When $0 < \kappa < (1/\pi)\ln|\alpha/\beta|$, $\epsilon_1^{(0)}(\lambda)$ has at most two zeros defined by $\epsilon_1^{(0)}(-B_1) = 0$ and $\epsilon_1^{(0)}(B_2) = 0$ where $B_2 \geqslant B_1 > 0$. Moreover, $\epsilon_1^{(0)+}(\lambda)$ is non-vanishing only in the interval $\lambda < -B_1$ or $\lambda > B_2$ in the ground state. Note that B_1 and B_2 approaches to infinity as H goes to zero. At strictly zero temperature the integral equation (3.1) can be written as

$$\epsilon_1^{(0)}(\lambda) = \tfrac{1}{2}H - 2\pi[\alpha G_0(\lambda-\kappa) + \beta G_0(\lambda+\kappa)] + G_1 * \epsilon_1^{(0)+}. \tag{3.2}$$

The free energy is then given by

$$\frac{F(0,H)}{2N} - \frac{F(0,0)}{2N} = -\frac{1}{2}\left(\int_{-\infty}^{-B_1} + \int_{B_2}^{\infty}\right)\mathrm{d}\lambda [G_0(\lambda-\kappa) + G_0(\lambda+\kappa)]\epsilon_1^{(0)}(\lambda). \tag{3.3}$$

Since equation (3.2) is not symmetric, unlike that for a single chain [14], it is convenient to introduce two functions $z(\lambda)$ and $y(\lambda)$ defined by $z(\lambda) = \epsilon_1^{(0)}(\lambda - B_1)$ and

$y(\lambda) = \epsilon_1^{(0)}(\lambda + B_2)$, then the ground-state integral equation (3.2) is rewritten as

$$z(\lambda) = \tfrac{1}{2}H - 2\pi[\alpha G_0(\lambda - B_1 - \kappa) + \beta G_0(\lambda - B_1 + \kappa)] + \int_{-\infty}^{0} \mathrm{d}\lambda'\, G_1(\lambda - \lambda')\, z(\lambda') + \int_0^{\infty} \mathrm{d}\lambda'\, G_1(\lambda - \lambda' - B_1 - B_2)\, y(\lambda') \tag{3.4}$$

or

$$y(\lambda) = \tfrac{1}{2}H - 2\pi[\alpha G_0(\lambda + B_2 - \kappa) + \beta G_0(\lambda + B_2 + \kappa)] + \int_0^{\infty} \mathrm{d}\lambda'\, G_1(\lambda - \lambda')\, y(\lambda') + \int_{-\infty}^{0} \mathrm{d}\lambda'\, G_1(\lambda - \lambda' + B_1 + B_2)\, z(\lambda'). \tag{3.5}$$

For a sufficiently weak magnetic field, i.e. $H \ll (\alpha + \beta)/(1 + \kappa^2)$, these coupled equations (3.4) and (3.5) can be solved using the Wiener–Hopf method [14] by iterations. Expanding $z(\lambda) = z_1(\lambda) + z_2(\lambda)$ and $y(\lambda) = y_1(\lambda) + y_2(\lambda)$ with $z_2(\lambda)$ and $y_2(\lambda)$ being of higher order in $1/(B_1 + B_2)$ than $z_1(\lambda)$ and $y_1(\lambda)$, respectively, we have

$$z_1(\lambda) - \int_{-\infty}^{0} \mathrm{d}\lambda'\, G_1(\lambda - \lambda')\, z_1(\lambda') = \tfrac{1}{2}H - 2\pi[\alpha G_0(\lambda - B_1 - \kappa) + \beta G_0(\lambda - B_1 + \kappa)] \tag{3.6}$$

$$y_1(\lambda) - \int_0^{\infty} \mathrm{d}\lambda'\, G_1(\lambda - \lambda')\, y_1(\lambda') = \tfrac{1}{2}H - 2\pi[\alpha G_0(\lambda + B_2 - \kappa) + \beta G_0(\lambda + B_2 + \kappa)] \tag{3.7}$$

$$z_2(\lambda) - \int_{-\infty}^{0} \mathrm{d}\lambda'\, G_1(\lambda - \lambda')\, z_2(\lambda') = \int_0^{\infty} \mathrm{d}\lambda'\, G_1(\lambda - \lambda' - B_1 - B_2)\, y_1(\lambda') \tag{3.8}$$

$$y_2(\lambda) - \int_0^{\infty} \mathrm{d}\lambda'\, G_1(\lambda - \lambda')\, y_2(\lambda') = \int_{-\infty}^{0} \mathrm{d}\lambda'\, G_1(\lambda - \lambda' + B_1 + B_2)\, z_1(\lambda'). \tag{3.9}$$

The solutions of these equations are the following:

$$\tilde{z}_1(\omega) = -\frac{\mathrm{i}H}{2}\frac{g(-\omega)\, g(0)}{\omega - \mathrm{i}0} + \mathrm{i}\pi \frac{g(-\omega)\, g\left(\frac{1}{2}\mathrm{i}\pi\right)}{\omega - \frac{1}{2}\mathrm{i}\pi}\left[\alpha \mathrm{e}^{-\frac{1}{2}\pi(B_1+\kappa)} + \beta \mathrm{e}^{-\frac{1}{2}\pi(B_1-\kappa)}\right] \tag{3.10}$$

$$\tilde{y}_1(\omega) = \frac{\mathrm{i}H}{2}\frac{g(\omega)\, g(0)}{\omega + \mathrm{i}0} - \mathrm{i}\pi \frac{g(\omega)\, g\left(\frac{1}{2}\mathrm{i}\pi\right)}{\omega + \frac{1}{2}\mathrm{i}\pi}\left[\alpha \mathrm{e}^{-\frac{1}{2}\pi(B_2-\kappa)} + \beta \mathrm{e}^{-\frac{1}{2}\pi(B_2+\kappa)}\right] \tag{3.11}$$

$$\tilde{z}_2(\omega) = -\frac{\mathrm{i}H}{2\pi}\frac{g(-\omega)\, g(0)}{\omega - \mathrm{i}0}\left(\frac{1}{B_1 + B_2} - \frac{2\ln(B_1 + B_2)}{\pi(B_1 + B_2)^2}\right) \tag{3.12}$$

$$\tilde{y}_2(\omega) = \frac{\mathrm{i}H}{2\pi}\frac{g(\omega)\, g(0)}{\omega + \mathrm{i}0}\left(\frac{1}{B_1 + B_2} - \frac{2\ln(B_1 + B_2)}{\pi(B_1 + B_2)^2}\right) \tag{3.13}$$

where

$$f(\lambda) = \frac{1}{2\pi}\int \mathrm{d}\omega\, \mathrm{e}^{-\mathrm{i}\omega\lambda}\, \tilde{f}(\omega) \qquad g(\omega) = \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2} - \mathrm{i}\omega/\pi\right)}\left(\frac{-\mathrm{i}\omega + 0}{\mathrm{e}\pi}\right)^{-\mathrm{i}\omega/\pi} \tag{3.14}$$

with Γ being a Gamma function. Note that these expressions of $z(\lambda)$ and $y(\lambda)$ are valid only for $\lambda < 0$ and $\lambda > 0$, respectively. Using the conditions $z(0) = 0$ and $y(0) = 0$, the following relations are now obtained, i.e.

$$B_1 \simeq -\frac{2}{\pi}\ln\left[\frac{H\,g(0)}{2\pi\,g\left(\frac{1}{2}\mathrm{i}\pi\right)\left(\alpha\mathrm{e}^{-\pi\kappa/2}+\beta\mathrm{e}^{\pi\kappa/2}\right)}\right] \tag{3.15}$$

and

$$B_2 \simeq -\frac{2}{\pi}\ln\left[\frac{H\,g(0)}{2\pi g\left(\frac{1}{2}\mathrm{i}\pi\right)\left(\alpha\mathrm{e}^{\pi\kappa/2}+\beta\mathrm{e}^{-\pi\kappa/2}\right)}\right]. \tag{3.16}$$

The field-dependent term in the free energy (3.3) is then rewritten as

$$\frac{F(0,H)}{2N}-\frac{F(0,0)}{2N} = -\int \mathrm{d}\omega\frac{\mathrm{e}^{\mathrm{i}\omega\kappa}+\mathrm{e}^{-\mathrm{i}\omega\kappa}}{8\pi\cosh\omega}\left[\mathrm{e}^{-\mathrm{i}\omega B_1}\,\tilde{z}(\omega)+\mathrm{e}^{\mathrm{i}\omega B_2}\,\tilde{y}(\omega)\right] \tag{3.17}$$

and hence the magnetic susceptibility becomes

$$\chi_s = \frac{1}{4\pi^2}\left(\frac{\mathrm{e}^{\frac{1}{2}\pi\kappa}+\mathrm{e}^{-\frac{1}{2}\pi\kappa}}{\alpha\mathrm{e}^{\frac{1}{2}\pi\kappa}+\beta\mathrm{e}^{-\frac{1}{2}\pi\kappa}}+\frac{\mathrm{e}^{-\frac{1}{2}\pi\kappa}+\mathrm{e}^{\frac{1}{2}\pi\kappa}}{\alpha\mathrm{e}^{-\frac{1}{2}\pi\kappa}+\beta\mathrm{e}^{\frac{1}{2}\pi\kappa}}\right)\left[1+\frac{1}{2|\ln H|}-\frac{\ln|\ln H|}{4|\ln H|^2}+\cdots\right]. \tag{3.18}$$

Note that in the limit $H \to 0$, the coupling between two spin chains affects only the magnitude in the susceptibility and the field-correction terms of the typical Heisenberg chain are preserved. The first logarithmic correction was anticipated by Yang and Yang [15] and calculated by Babujian [14], while the next-to-leading logarithmic contribution was obtained by Lee and Schlottmann [16]. Of course, the magnetic susceptibility for $\kappa = 0$ reduces to the well known value $1/\left[\pi^2(\alpha+\beta)\right]$ of the Heisenberg chain with an antiferromagnetic coupling $(\alpha+\beta)$ [17, 18].

3.2. Linear specific heat coefficient γ in field

The low-temperature specific heat is linear in temperature. To obtain the linear coefficient γ in the low-temperature expansion of the free energy, we need the T^2-correction terms to $\epsilon_1(\lambda)$, i.e.

$$\epsilon_1(\lambda) \simeq \epsilon_1^{(0)}(\lambda)+T^2\,\epsilon_1^{(2)}(\lambda). \tag{3.19}$$

The ground-state integral equation (3.1) is then separated by using a Sommerfeld expansion,

$$T\ln\left(1+\mathrm{e}^{\epsilon_1/T}\right) \simeq \epsilon_1^{+}(\lambda)+\tfrac{1}{6}(\pi T)^2\delta(\epsilon_1). \tag{3.20}$$

$\epsilon_1^{(0)}$ satisfies the integral equation (3.2) and $\epsilon_1^{(2)}$ responsible for the specific heat satisfies the following integral equation:

$$\begin{aligned}\epsilon_1^{(2)}(\lambda) = \frac{\pi^2}{6}\Bigg[&\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|^{-1}_{-B_1} G_1(\lambda+B_1)+\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|^{-1}_{B_2} G_1(\lambda-B_2)\Bigg]\\ &+\left(\int_{-\infty}^{-B_1}+\int_{B_2}^{\infty}\right)\mathrm{d}\lambda'\,G_1(\lambda-\lambda')\,\epsilon_1^{(2)}(\lambda').\end{aligned} \tag{3.21}$$

Considering the $\epsilon_1^{(2)}$ contribution to the free energy, the linear coefficient γ of the specific heat can be expressed by

$$\gamma = \frac{\pi^2}{6}\Bigg[\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|_{-B_1}^{-1}\Big\{G_0(B_1-\kappa)+G_0(B_1+\kappa)\Big\} + \left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|_{B_2}^{-1}\{G_0(B_2-\kappa)+G_0(B_2+\kappa)\}\Bigg] + \left(\int_{-\infty}^{-B_1}+\int_{B_2}^{\infty}\right)\mathrm{d}\lambda\,[G_0(\lambda-\kappa)+G_0(\lambda+\kappa)]\,\epsilon_1^{(2)}(\lambda). \tag{3.22}$$

Hence, to calculate γ, it is necessary to solve the integral equation of $\epsilon_1^{(2)}(\lambda)$ in (3.21). Our procedure to solve (3.21) is similar to the one used to obtain $\epsilon_1^{(0)}$ in the above. Defining $\psi(\lambda) = \epsilon_1^{(2)}(\lambda - B_1)$ and $\varphi(\lambda) = \epsilon_1^{(2)}(\lambda + B_2)$, the integral equation (3.21) is reduced to two coupled integral equations for $\psi(\lambda)$ and $\varphi(\lambda)$,

$$\psi(\lambda) = \frac{\pi^2}{6}\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|_{-B_1}^{-1} G_1(\lambda) + \frac{\pi^2}{6}\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|_{B_2}^{-1} G_1(\lambda - B_1 - B_2) + \int_{-\infty}^{0}\mathrm{d}\lambda'\,G_1(\lambda-\lambda')\,\psi(\lambda') + \int_0^{\infty}\mathrm{d}\lambda'\,G_1(\lambda-\lambda'-B_1-B_2)\,\varphi(\lambda') \tag{3.23}$$

$$\varphi(\lambda) = \frac{\pi^2}{6}\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|_{B_2}^{-1} G_1(\lambda) + \frac{\pi^2}{6}\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|_{-B_1}^{-1} G_1(\lambda + B_1 + B_2) + \int_0^{\infty}\mathrm{d}\lambda'\,G_1(\lambda-\lambda')\,\varphi(\lambda') + \int_{-\infty}^{0}\mathrm{d}\lambda'\,G_1(\lambda-\lambda'+B_1+B_2)\,\varphi(\lambda'). \tag{3.24}$$

The absence of symmetry in $\epsilon_1^{(2)}(\lambda)$ again leads to the coupled forms of these integral equations. Considering the leading contribution since the next terms can only contribute to order $1/(B_1+B_2)^2$ or higher in the free energy, equations (3.23) and (3.24) are simply rewritten as

$$\psi(\lambda) = \frac{\pi^2}{6}\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|_{-B_1}^{-1} G_1(\lambda) + \int_{-\infty}^{0}\mathrm{d}\lambda'\,G_1(\lambda-\lambda')\,\psi(\lambda') \tag{3.25}$$

$$\varphi(\lambda) = \frac{\pi^2}{6}\left|\frac{\mathrm{d}\epsilon_1^{(0)}}{\mathrm{d}\lambda}\right|_{B_2}^{-1} G_1(\lambda) + \int_0^{\infty}\mathrm{d}\lambda'\,G_1(\lambda-\lambda')\,\varphi(\lambda'). \tag{3.26}$$

These equations are the same as those for the single chain [19] except for the driving terms. The driving terms are now changed since the integral equation of $\epsilon_1^{(0)}$ in (3.2) has been changed with parameters α, β and κ. Hence

$$\left|\frac{\mathrm{d}\epsilon_1^{(0)}(\lambda)}{\mathrm{d}\lambda}\right|_{-B_1} = \tfrac{1}{2}\pi^2 g\left(\tfrac{1}{2}\mathrm{i}\pi\right)\mathrm{e}^{-\frac{1}{2}\pi B_1}\left(\alpha\mathrm{e}^{-\frac{1}{2}\pi\kappa}+\beta\mathrm{e}^{\frac{1}{2}\pi\kappa}\right) \tag{3.27}$$

$$\left|\frac{\mathrm{d}\epsilon_1^{(0)}(\lambda)}{\mathrm{d}\lambda}\right|_{B_2} = \tfrac{1}{2}\pi^2 g\left(\tfrac{1}{2}\mathrm{i}\pi\right)\mathrm{e}^{-\frac{1}{2}\pi B_2}\left(\alpha\mathrm{e}^{\frac{1}{2}\pi\kappa}+\beta\mathrm{e}^{-\frac{1}{2}\pi\kappa}\right). \tag{3.28}$$

The integral equations for ψ and φ are of the Wiener–Hopf type and the solutions $\psi(\lambda)$ for $\lambda \leqslant 0$ and $\varphi(\lambda)$ for $\lambda \geqslant 0$ can be obtained as

$$\psi(\lambda) = \frac{1}{3\sqrt{2}H}\int \mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}\omega\lambda}[g(-\omega)-1] \tag{3.29}$$

$$\varphi(\lambda) = \frac{1}{3\sqrt{2}\,H}\int \mathrm{d}\omega\,\mathrm{e}^{-\mathrm{i}\omega\lambda}[g(\omega)-1]. \tag{3.30}$$

Inserting ψ and φ into the γ expression in (3.22), we obtain that

$$\gamma = \frac{1}{6}\left[\frac{e^{\frac{1}{2}\pi\kappa} + e^{-\frac{1}{2}\pi\kappa}}{\alpha e^{\frac{1}{2}\pi\kappa} + \beta e^{-\frac{1}{2}\pi\kappa}} + \frac{e^{-\frac{1}{2}\pi\kappa} + e^{\frac{1}{2}\pi\kappa}}{\alpha e^{-\frac{1}{2}\pi\kappa} + \beta e^{\frac{1}{2}\pi\kappa}}\right] \tag{3.31}$$

in a weak magnetic field. This result correctly reduces to $2/[3(\alpha+\beta)]$ for $\kappa = 0$.

4. Low-temperature limit in zero field

So far we have considered the limit of $T \to 0$ and $0 < H \ll (\alpha+\beta)/(1+\kappa^2)$ in the previous section and have shown that only one integral equation for $n = 1$ plays a significant role. On the other hand, the case when $0 < T \ll (\alpha+\beta)/(1+\kappa^2)$ and $H = 0$ is also interesting. In this case, however, we have to consider the infinitely coupled integral equations (2.10). Fortunately, we can calculate the entropy for this limit by the method used by Babujian [14] and Filyov *et al* [20]. Note that we are still assuming the limit of $0 < \kappa < (1/\pi)\ln|\alpha/\beta|$ for the analytic approach.

The integral equations in (2.5) for the distribution functions and (2.10) for the energy potentials can be rewritten as

$$\rho_n(\lambda) + \rho_{n,h}(\lambda) = \tfrac{1}{2}[G_0(\lambda-\kappa) + G_0(\lambda+\kappa)]\delta_{n,1} + G_0 * [\rho_{n-1,h} + \rho_{n+1,h}] \tag{4.1}$$

$$\epsilon_n(\lambda) = -2\pi[\alpha G_0(\lambda-\kappa) + \beta G_0(\lambda+\kappa)]\delta_{n,1} + TG_0 * \ln\left(1+e^{\epsilon_{n-1}/T}\right)\left(1+e^{\epsilon_{n+1}/T}\right) \tag{4.2}$$

respectively. It is important to note that these integral equations have the same integral kernels and the similar driving terms to each other. Due to the asymmetry of $\epsilon_n(\lambda)$, let us consider these equations in two parts, i.e. $\lambda > 0$ and $\lambda < 0$. Substituting $\lambda \to \Lambda - (2/\pi)\ln(2T/\pi)$ for $\lambda > 0$ and $\lambda \to \Lambda + 2/\pi\ln(2T/\pi)$ for $\lambda < 0$ in these equations, and then differentiating (4.2) with respect to Λ, we obtain interesting relations between the energy potentials and the density functions, i.e.

$$\begin{aligned}\rho_n(\lambda) &= \frac{1}{2\pi^2}\mathcal{C}(\alpha,\beta,\pm\kappa)\frac{d\epsilon_n}{d\Lambda}f(\epsilon_n)\\ \rho_{n,h}(\lambda) &= \frac{1}{2\pi^2}\mathcal{C}(\alpha,\beta,\pm\kappa)\frac{d\epsilon_n}{d\Lambda}[1-f(\epsilon_n)]\end{aligned} \tag{4.3}$$

where the upper (lower) sign is for $\lambda > 0$ ($\lambda < 0$), and we have used $f(\epsilon_n) = \left(1+e^{\epsilon_n/T}\right)^{-1}$ and

$$\mathcal{C}(\alpha,\beta,\kappa) = \left(e^{\frac{1}{2}\pi\kappa} + e^{-\frac{1}{2}\pi\kappa}\right)\big/\left(\alpha e^{\frac{1}{2}\pi\kappa} + \beta e^{-\frac{1}{2}\pi\kappa}\right). \tag{4.4}$$

These relations are only valid for sufficiently low T. For convenience, defining $E_n^{\pm}(\Lambda) = \epsilon_n[\lambda \mp (2/\pi)\ln(2T/\pi)]/T$ and substituting the relations in (4.3) into the entropy in (2.8), we have

$$\begin{aligned}\frac{S}{2N} = -\frac{T}{2\pi^2}\sum_{\sigma=\pm}\mathcal{C}(\alpha,\beta,\sigma\kappa)\sum_{n=1}^{\infty}\int_{E_n^{\sigma,\min}}^{E_n^{\sigma,\max}} dE_n^{\sigma}\,&\left\{f\left(TE_n^{\sigma}\right)\ln f\left(TE_n^{\sigma}\right)\right.\\ &\left.+\left[1-f\left(TE_n^{\sigma}\right)\right]\ln\left[1-f\left(TE_n^{\sigma}\right)\right]\right\}\end{aligned} \tag{4.5}$$

where $E_n^{\pm}$ satisfy the following integral equations:

$$E_n^{\pm}(\Lambda) = -\left(\alpha e^{\pm\frac{1}{2}\pi\kappa} + \beta e^{\mp\frac{1}{2}\pi\kappa}\right)e^{\mp\frac{1}{2}\pi\Lambda}\delta_{n,1} + G_0 * \ln\left(1+e^{E_{n-1}^{\pm}}\right)\left(1+e^{E_{n+1}^{\pm}}\right) \tag{4.6}$$

with the asymptotic condition $\lim_{n\to\infty}\left(E_n^{\pm}/n\right) = 0$. Since the driving term is negative, $E_n^{\pm\max} = E_n^{\pm}(\pm\infty)$ and $E_n^{\pm\min} = E_n^{\pm}(\mp\infty)$. Hence the integral equations for maximum or minimum solutions are reduced to the difference equations such that

$$
\begin{aligned}
E_n^{\pm\max} &= \tfrac{1}{2}\ln\left(1+\mathrm{e}^{E_{n-1}^{\pm\max}}\right)\left(1+\mathrm{e}^{E_{n+1}^{\pm\max}}\right)\\
E_{n\neq 1}^{\pm\min} &= \tfrac{1}{2}\ln\left(1+\mathrm{e}^{E_{n-1}^{\pm\min}}\right)\left(1+\mathrm{e}^{E_{n+1}^{\pm\min}}\right)\\
E_{n=1}^{\pm\min} &= -\infty.
\end{aligned}
\tag{4.7}
$$

The solutions of these equations are of the form [14]

$$
\begin{aligned}
E_n^{\pm\max} &= \ln\left[(n+1)^2-1\right]\\
E_n^{\pm\min} &= \ln\left(n^2-1\right).
\end{aligned}
\tag{4.8}
$$

The entropy (4.5) in zero magnetic field is then expressed as

$$
\frac{S}{2N} = -\frac{T}{2\pi^2}[\mathcal{C}(\alpha,\beta,\kappa)+\mathcal{C}(\alpha,\beta,-\kappa)]\int_0^1 \mathrm{d}x\left[\frac{\ln x}{1-x}+\frac{\ln(1-x)}{x}\right]. \tag{4.9}
$$

The integral is the dilogarithmic function yielding $-\pi^2/3$, and so the linear coefficient γ of specific heat in zero magnetic field,

$$
\gamma = \frac{1}{6}\left(\frac{\mathrm{e}^{\frac{1}{2}\pi\kappa}+\mathrm{e}^{-\frac{1}{2}\pi\kappa}}{\alpha\mathrm{e}^{\frac{1}{2}\pi\kappa}+\beta\mathrm{e}^{-\frac{1}{2}\pi\kappa}}+\frac{\mathrm{e}^{-\frac{1}{2}\pi\kappa}+\mathrm{e}^{\frac{1}{2}\pi\kappa}}{\alpha\mathrm{e}^{-\frac{1}{2}\pi\kappa}+\beta\mathrm{e}^{\frac{1}{2}\pi\kappa}}\right) \tag{4.10}
$$

which agrees with the γ in a finite field obtained in the previous section. Hence the limits $\lim_{T\to 0}$ and $\lim_{H\to 0}$ commute with each other in the low-temperature specific heat.

5. Concluding remarks

We considered an exactly solvable two-chain quantum spin-$\frac{1}{2}$ model in generalized form which enhances the spin frustration of the Hamiltonian. The transfer matrix of the system was constructed by using the quantum inverse scattering method. This transfer matrix modifies the one for a simple two-chain quantum spin model [7–9, 11] with two arbitrary constants α and β such as $\hat{T}(\lambda) = \hat{t}^{\alpha}(\lambda+\kappa)\,\hat{t}^{\beta}(\lambda)$. $\hat{t}$ is the typical transfer matrix of the spin-$\frac{1}{2}$ Heisenberg quantum chain. We obtained the Hamiltonian from the logarithmic derivative of the transfer matrix. The Hamiltonian is more frustrated by two constants α, which is related to the second chain, and β, which is related to the first chain. To consider the antiferromagnetic coupling between the chains, we limited ourselves to the case of $\alpha > 0$, $\beta < 0$ and $\alpha+\beta > 0$. The Hamiltonian has an unusual term, the so-called chiral term [7–9, 11], since this unusual term breaks both T and P symmetries while conserving TP symmetry when $\alpha = \beta$.

By construction, the eigenfunctions of the system do not change with α and β, but the energy changes. We constructed the thermodynamic Beth ansatz equations which differ by the driving terms from the simple two-chain model [7–9, 11]. The difference of driving terms resulted in the asymmetry of the energy potentials. To obtain the thermodynamic properties, we considered two limits which can be treated analytically, namely, (i) $T \to 0$ and $0 < H \ll (\alpha+\beta)/(1+\kappa^2)$ and (ii) $H \to 0$ and $0 < T \ll (\alpha+\beta)/(1+\kappa^2)$ with the restriction that $\kappa < (1/\pi)\ln|\alpha/\beta|$.

For the limit (i), we calculated the magnetic susceptibility and the specific heat using the ground-state thermodynamic Bethe ansatz equation. We found that the constants α and β contribute rather complicatedly to the amplitudes of the magnetic susceptibility and the specific heat with the same factor $\left(\mathrm{e}^{\frac{1}{2}\pi\kappa}+\mathrm{e}^{-\frac{1}{2}\pi\kappa}\right)\left[\left(\alpha\mathrm{e}^{\frac{1}{2}\pi\kappa}+\beta\mathrm{e}^{-\frac{1}{2}\pi\kappa}\right)^{-1}+\left(\alpha\mathrm{e}^{-\frac{1}{2}\pi\kappa}+\beta\mathrm{e}^{\frac{1}{2}\pi\kappa}\right)^{-1}\right]$. The spin excitations show no gap even though the enhanced spin frustration has been considered. The reason for this might be due to the third term in the Hamiltonian [9].

For the limit (ii), we had to consider all excitations to see the thermodynamic behaviour since the temperature is finite. Following Babujian's procedure [14], the entropy of the system can be calculated. The specific heat in this limit coincided to the result of the limit (i). This implies that the two limits, $\lim_{T\to 0}$ and $\lim_{H\to 0}$ are commuting to each other. Hence the Wilson ratio $\lim_{T\to 0} \gamma/\chi = 2\pi^2/3$ is universal as in the Kondo problem. In this paper we have restricted the coupling constant κ to $0 < \kappa < (1/\pi) \ln|\alpha/\beta|$ for the analytic discussions. The large-coupling limit, however, is also interesting and expected to provide some new behaviours. The detailed work will be published elsewhere.

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